Asymptotic modelling of a fluid-structure coupling in the case of a prestressed inflated orthotropic membrane shell

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Abstract

In this Note, we consider an inflated orthotropic linearly elastic generalized membrane shell submited to an outer surface perturbation. We obtain the strong convergence towards the solution of a wellposed "2D" problem of the mean value in the membrane thickness 2ε of the "3D" scaled displacements, as ε approaches zero.

Résumé

Modélisation asymptotique d'un problème de couplage fluide-structure dans le cas d'une membrane orthotrope gonflée précontrainte. Dans cette Note, on considère une coque en membrane en flexion pure inhibée, gonflée et précontrainte, que l'on soumet à des perturbations extérieures. On établit, quand l'épaisseur 2ε de la coque tend vers zéro, la convergence forte de la valeur moyenne dans l'épaisseur du déplacement "3D" normalisé vers la solution d'un problème "2D" bien posé.

1 Version française abrégée

On considère une structure souple gonflée précontrainte composée de deux pièces d'un tissu orthotrope fixées sur une armature rigide. Le volume intérieur V ainsi délimité est gonflé sous pression Π par un gaz parfait, créant ainsi une précontrainte, supposée connue, $\mathbf{\Sigma_0}$. On applique à ce système en équilibre une perturbation surfacique extérieure $\mathbf{h}$. Le fluide intérieur génère un couplage entre les déplacements, et donc entre les contraintes, des deux pièces de tissu. On suppose que du fait de la précontrainte, la déformation se fait à volume constant et qu'on reste dans le cadre de l'élasticité linéaire. On suppose également que la géométrie et les conditions d'encastrement sont telles que le matériau se comporte comme une coque en membrane généralisée linéairement élastique de type 1, c'est à dire en flexion pure inhibée (voir définition dans [Ciarlet, 2000] p262 ou dans [Sanchez-Hubert, Sanchez-Palencia, 1997]). Enfin, on suppose que la densité $\mathbf{h}$ est admissible au sens donné dans [Luce-Poutous-Thomas, 2007] et que la forme linéaire liée au couplage est continue par rapport à la norme membranaire.

Soit un domaine $\omega = \omega^+ \cup \omega^-$ de $\mathbb{R}^2$, de paramétrisation de la surface moyenne des tissus $S = S^+ \cup S^-$ en coordonnées curvilignes choisies le long des directions principales d'orthotropie. On considère les cartes associées respectives θ^+ et θ^- ($\theta^\pm : \overline{\omega^\pm} \longrightarrow \mathbb{R}^3 \in C^3(\overline{\omega^\pm}; \mathbb{R}^3)$) injectives, telles que, pour $\alpha \in \{1, 2\}$, les vecteurs $a_\alpha^\pm = \partial_\alpha \theta^\pm(y)$ sont orthogonaux en tout point $y \in \overline{\omega^\pm}$. On leur associe la famille de coques de surface moyenne $S = \theta(\overline{\omega})$, et d'épaisseur 2ε. L'encastrement se traduit par des conditions aux limites de type Dirichlet homogène posées sur une même portion du bord latéral $\theta^\pm\left(\gamma_0^\pm\right)$. En faisant tendre l'épaisseur vers zéro, on se ramène à un problème variationnel posé sur S, bien posé puisque $\mathbf{h}$ appartient au dual de l'espace où on cherche les déplacements (voir [Chapelle-Bathe, 2003], p129 ou [Ciarlet, 2000]).

En partant de la démonstration donnée dans [Ciarlet-Lods, 1996], on élargit les résultats de convergences fortes au cas d'un matériau orthotrope avec un couplage fluide structure. On prouve que la solution $\mathbf{u}(\varepsilon)$ de (1), problème variationnel normalisé 3D associé où figure le terme de couplage, converge fortement dans un espace obtenu par complétion, et que sa valeur moyenne dans l'épaisseur, $\overline{\mathbf{u}(\varepsilon)}$, converge fortement également, dans un espace complété qui prend en compte le couplage, vers l'unique solution ζ du problème variationnel (2) posé cette fois-ci sur une surface. Dans le modèle asymptotique obtenu, l'équation variationnelle est identique à celle que l'on obtiendrait sans le couplage, mais l'espace dans lequel on cherche la solution est un sous-espace fermé de l'espace naturel pour les coques en membranes.

2 Notations and 3D modelling

In this Note, greek indices take their values in $\{1,2\}$, whereas latin indices belong to $\{1,2,3\}$, the repeated index summation convention is used and δ^i_j is the Kronecker symbol.

Assumption 1 *Let $\omega^\pm$ be domains in $\mathbb{R}^2$ (open, bounded, connected subsets with a Lipschitz-continuous boundary, the sets $\omega^\pm$ being locally on one side of their boundary), let $\theta^\pm : \overline{\omega^\pm} \longrightarrow \mathbb{R}^3 \in C^3\left(\overline{\omega};\mathbb{R}^3\right)$ be injective mappings such that the vectors $\mathbf{a}^\pm{}_\alpha := \partial_\alpha \theta^\pm(y)$ are orthogonal at each point $y \in \overline{\omega^\pm}$. Let the surfaces $S^\pm := \theta^\pm(\overline{\omega^\pm})$ represent the middle surfaces of two pieces of an orthotropic material which orthotropy directions are $\mathbf{a}^\pm{}_\alpha$ in the surface and $\mathbf{a}^\pm{}_3 := \frac{\mathbf{a}_1^\pm \wedge \mathbf{a}_2^\pm}{|\mathbf{a}_1^\pm \wedge \mathbf{a}_2^\pm|}$ in the thickness. Let 2ε be the thickness of the material and $\Omega^\pm_\varepsilon := S^\pm \times]-\varepsilon,\varepsilon[$ be the volume occupied by the shells. Let $S^\pm_\varepsilon := S^\pm \times \{\varepsilon\}$ be the outer surfaces and $S^\pm_{-\varepsilon} := S^\pm \times \{-\varepsilon\}$ be the inner surfaces. Let $\partial\omega^\pm = \gamma^\pm_0 \cup \gamma^\pm_1$, $\Gamma^\pm_{0,\varepsilon} := \theta^\pm(\gamma^\pm_0) \times]-\varepsilon,\varepsilon[$ and $\Gamma^\pm_{1,\varepsilon} := \theta^\pm(\gamma^\pm_1) \times]-\varepsilon,\varepsilon[$. We suppose that the shells are clamped along $\Gamma^\pm_{0,\varepsilon}$ in a prestressed equilibrium state, that the inflating pressure is Π, the prestress $\mathbf{\Sigma}^\pm_0$.*

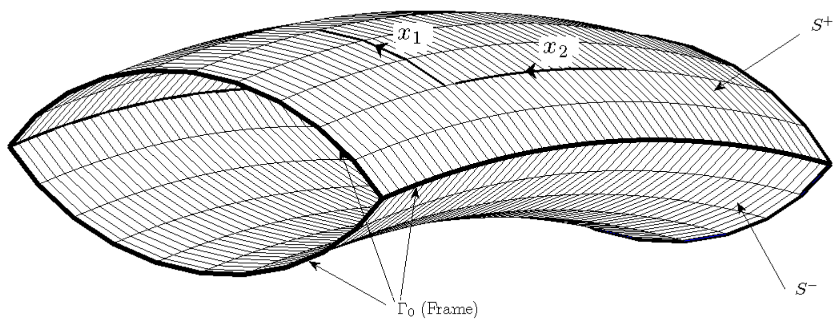

Figure 1: an example of prestressed system

Remark 1 *Let us insist on the fact that the orthotropy of the material imposes the choice of the surface parameters contrary to the usual isotropic case where the choice is free.*

We pertubate the outer conditions with a surface density force $\mathbf{h} \in L^2(S^\pm_{-\varepsilon})$. This perturbation modifies the geometry of the shells and thereby the inner pressure of $\Delta\Pi$ and the stresses which become $\mathbf{\Sigma}^\pm$. We let $\mathbf{n}$ denote the outer unit normal vector to the current surface and $\mathbf{A}$ denote the elasticity tensor in an orthotropy orthonormal base (this is the natural base to underline the symetry properties of the tensor and to cancel most of the coordinates). Its contravariant components are

$$A^{ijkl} = 0 \text{ apart from } A^{iijj} = \lambda_{ij} \text{ and } A^{ijij} = A^{ijji} = \mu_{ij} \text{ when } i \neq j$$

where $\lambda_{ij} = \lambda_{ji}$ and $\mu_{ij} = \mu_{ji}$ are positive constants which only depend on the material, (see [Coirier, 2001] for the calculus). The equilibrium equations in the reference configuration (prestressed state) are :

$$\begin{cases} -\mathbf{div}(\mathbf{\Sigma}^\pm - \mathbf{\Sigma}^\pm_0) = \mathbf{0} & \text{in } \Omega^\pm_\varepsilon \\ (\mathbf{\Sigma}^\pm - \mathbf{\Sigma}^\pm_0)\,\mathbf{n} = \mathbf{h} & \text{on } S^\pm_\varepsilon \\ (\mathbf{\Sigma}^\pm - \mathbf{\Sigma}^\pm_0)\,\mathbf{n} = -\Delta\Pi\,\mathbf{n} & \text{on } S^\pm_{-\varepsilon} \\ (\mathbf{\Sigma}^\pm - \mathbf{\Sigma}^\pm_0)\,\mathbf{n} = \mathbf{0} & \text{on } \Gamma^\pm_{1,\varepsilon}. \end{cases}$$

Assumption 2 *Elasticity laws can be linearized (expected effect of the prestress).*

Let $\mathbf{u}_\varepsilon^\pm$ denote the displacement between the prestressed configuration and the deformed configuration. Then,

$$\mathbf{\Sigma}^\pm - \mathbf{\Sigma}_0^\pm = \mathbf{A} : \mathbf{e}_\varepsilon^\pm(\mathbf{u}_\varepsilon^\pm) \quad where \quad \mathbf{e}_\varepsilon^\pm(\mathbf{u}_\varepsilon^\pm) = \frac{1}{2}(\nabla \mathbf{u}_\varepsilon^\pm + \nabla \mathbf{u}_\varepsilon^{\pm T}).$$

The inflating gas is perfect so the inner volume variation ΔV and the inner pressure variation $\Delta\Pi$ are linked by $\Delta\Pi = -\frac{\Delta V}{V}\Pi$. The volume variation and the displacement field satisfy $\Delta V = -\int_{S^\pm_{-\varepsilon}} \mathbf{u}_\varepsilon^\pm \cdot \mathbf{n}\, ds$. Lastly, the clamping condition reads $\mathbf{u}_\varepsilon^\pm = \mathbf{0}$ on $\Gamma^\pm_{0,\varepsilon}$.

Since the unknowns are defined on sets that vary with ε, we naturally transform the 3D variational problems equivalent to the former PDE and boundary conditions system into problems posed over a set that does not depend on ε through the bijection $(y, x_3) \longmapsto (y, \varepsilon x_3)$ and thus obtain : $\mathbf{u}(\varepsilon) \in \mathbf{V}(\Omega), \forall \mathbf{v} \in \mathbf{V}(\Omega)$

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon; \mathbf{u}(\varepsilon)) e_{i\|j}(\varepsilon; \mathbf{v}) \sqrt{g(\varepsilon)}\, dx + \frac{1}{\varepsilon}\int_{\Gamma^-} \mathbf{u}(\varepsilon)\cdot\mathbf{n}\,\sqrt{g(\varepsilon)} dy \int_{\Gamma^-} \mathbf{v}\cdot\mathbf{n}\,\sqrt{g(\varepsilon)} dy = \int_{\Gamma^+} h^{i\pm} v_i \sqrt{g(\varepsilon}\, dy \qquad (1)$$

The functions $h^{i\pm}$ are in $L^2(\Gamma^+)$ and independent of ε, $\Gamma^+ := \omega\times\{1\}$, $\Gamma^- := \omega\times\{-1\}$, $\Omega := \omega\times]-1,1[$, $\Gamma_0 = \gamma_0\times]-1,1[$ and $\mathbf{V}(\Omega) := \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}$, $\mathbf{\Theta} : \overline{\omega}\times[-\varepsilon,\varepsilon] \longrightarrow \overline{\Omega_\varepsilon^\pm}$ is the canonical extension of θ and thus verifies $\mathbf{\Theta}(y, x_3) := \theta(y) + x_3 a_3$ and $\det(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) > 0$ ($(\mathbf{g}_i)_i := (\partial_i\mathbf{\Theta})_i$ is the covariant base of the set $\mathbf{\Theta}(\overline{\omega}\times[-\varepsilon,\varepsilon])$ whereas $(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)$ is the contravariant base such that $\mathbf{g}^j\cdot\mathbf{g}_i = \delta_i^j$). Then, $g(\varepsilon)$ denotes the scaled function of $g^\varepsilon := \det(\mathbf{g}_i\cdot\mathbf{g}_j)_{ij}$ and $A^{ijkl}(\varepsilon)$ are the contravariant components of the scaled 3D elasticity tensor $\mathbf{A}(\varepsilon)$. They satisfy the change of bases and coordinates formula

$$A^{ijkl}(\varepsilon)(y, x_3) = (|\mathbf{g}^i|\,|\mathbf{g}^j|\,|\mathbf{g}^k|\,|\mathbf{g}^l|\,A^{ijkl})(y, \varepsilon x_3),$$

the symetry conditions $A^{ijkl}(\varepsilon) = A^{jikl}(\varepsilon) = A^{klij}(\varepsilon)$ and the first order developments $A^{ijkl}(\varepsilon) = A^{ijkl}(0) + O(\varepsilon)$. The order symbol is meant with respect to the norm $\|w\|_{0,\infty,\overline{\Omega}} := \sup\left\{|w(x)|, x \in \overline{\Omega}\right\}$. Moreover,

$$A^{\alpha\beta\sigma3}(\varepsilon) = A^{\alpha333}(\varepsilon) = A^{\alpha\beta\sigma3}(0) = A^{\alpha333}(0) = 0.$$

For any vector field $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, the scaled linearized strains $e_{i\|j}(\varepsilon; \mathbf{v}) = e_{j\|i}(\varepsilon; \mathbf{v}) \in \mathbf{L}^2(\Omega)$ are defined by $e_{\alpha\|\beta}(\varepsilon; \mathbf{v}) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p$, $\quad e_{\alpha\|3}(\varepsilon; \mathbf{v}) := \frac{1}{2}(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3) - \Gamma^\sigma_{\alpha3}(\varepsilon) v_\sigma$ and $e_{3\|3}(\varepsilon; \mathbf{v}) := \frac{1}{\varepsilon}\partial_3 v_3$ with $\Gamma^p_{ij}(\varepsilon)$ being the scaled 3D Christoffel symbols obtained from $\Gamma^{\varepsilon,p}_{ij} := \mathbf{g}^p\cdot\partial_i\mathbf{g}_j$.

Proposition 1 *The scaled variational problems (1) are well posed.*

Proof. The bilinear form $B_\varepsilon(\mathbf{u}, \mathbf{v}) := \int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon; \mathbf{u}) e_{i\|j}(\varepsilon; \mathbf{v}) \sqrt{g(\varepsilon)}\, dx$ is a norm over $\mathbf{V}(\Omega)$ equivalent (inequality of Korn) to the usual norm. The coupling term $C_\varepsilon(\mathbf{u}, \mathbf{v}) := \frac{1}{\varepsilon}\int_{\Gamma^-} u_3\,\sqrt{g(\varepsilon)} dy \int_{\Gamma^-} v_3\,\sqrt{g(\varepsilon)} dy$ is bicontinuous and increases the positivity of the operator. The second member $L_\varepsilon(\mathbf{u}) := \int_{\Gamma^+} h^{i\pm} v_i \sqrt{g(\varepsilon)}\, dy$ is continuous. ■

In this Note, we establish that the coupling term remains bounded and we prove the strong convergence of the sequences $(\mathbf{u}(\varepsilon))_\varepsilon$ and $(\overline{\mathbf{u}(\varepsilon)})_\varepsilon := (\frac{1}{2}\int_{-1}^1 \mathbf{u}(\varepsilon)\, dx_3)_\varepsilon$ as ε approaches zero. The asymptotic 2D model is very close to the orthotropic membrane model obtained without the coupling.

3 Main results

In a computing perspective, we are interested in the convergence of $(\overline{\mathbf{u}(\varepsilon)})_\varepsilon$ towards the solution of a variational problem posed over a surface. Analytical preliminaries (we let ε approach zero in (1) when $\mathbf{v} = \varepsilon\mathbf{w}$ and $\mathbf{w}$ is an arbitrary function in the space $\mathbf{V}(\Omega)$) suggest to consider the 2D elasticity tensor $\mathbf{A}(\omega)$ which contravariant components are

$$a^{\alpha\beta\sigma\tau} := 2|\mathbf{a}^\alpha|\,|\mathbf{a}^\beta|\,|\mathbf{a}^\sigma|\,|\mathbf{a}^\tau|\left(A^{\alpha\beta\sigma\tau} - \frac{A^{\alpha\beta33}A^{33\sigma\tau}}{A^{3333}}\right)$$

and to consider, for any vector field (η_i) in $\mathbf{V}(\omega) := \{\eta = (\eta_i) \in \mathbf{H}^1(\omega); \eta = \mathbf{0} \text{ on } \gamma_0\}$, the 2D linearized change of metric tensor of covariant components $\gamma_{\alpha\beta}(\eta) := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$ where $b_{\alpha\beta} := \mathbf{a}_3 \cdot \partial_\alpha \mathbf{a}_\beta$ are the covariant components of the curvature tensor and $\Gamma^\sigma_{\alpha\beta} := \mathbf{a}^\sigma \cdot \partial_\alpha \mathbf{a}_\beta$ are the surface Christoffel symbols. Let us introduce the semi-norm $|\cdot|^M_\omega$ defined by $|\eta|^M_\omega := \{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\eta)|^2_{0,\omega}\}^{1/2}$.

Remark 2 *At this step, let us underline that the orthotropy does not affect the curvature tensor nor the semi-norm, it only changes the elasticity tensor.*

Assumption 3 *The geometry and the boundary conditions are such that $|\cdot|^M_\omega$ is a norm over the space $\mathbf{V}(\omega)$ which is not equivalent to the norm $\|\cdot\|_{1,\omega}$.*

Hence we let $\mathbf{V}^\#_M(\omega)$ be the completion of $\mathbf{V}(\omega)$ with respect to $|\cdot|^M_\omega$ and we associate to $|\cdot|^M_\omega$ the 3D norm over $\mathbf{V}(\Omega)$ defined by $|\mathbf{v}|^M_\Omega = \{|\partial_3 \mathbf{v}|^2_{0,\omega} + (|\overline{\mathbf{v}}|^M_\omega)^2\}^{1/2}$. The completion of $\mathbf{V}(\Omega)$ with respect to $|\cdot|_\Omega$ is $\mathbf{V}^\#_M(\Omega)$. Last, we let $a_{\alpha\beta} := \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ be the covariant components of the metric tensor and $a := det(a_{\alpha\beta})$. The bilinear form $B_M(\zeta, \eta) := \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\zeta)\gamma_{\alpha\beta}(\eta)\sqrt{a}dy$ is obviously continuous with respect to $|\cdot|^M_\omega$ so we let $B^\#_M$ denote the unique continuous extension of the bilinear form B_M from $\mathbf{V}(\omega)$ to $\mathbf{V}^\#_M(\omega)$. When $L_M(\eta) := \int_\omega h^i \eta_i \sqrt{a}\, dy$ and $C_M(\eta) := \int_\omega \eta_3 \sqrt{a}\, dy$ are $|\cdot|^M_\omega$ -continuous, we let $L^\#_M$ and $C^\#_M$ be their unique continuous extension from $\mathbf{V}(\omega)$ to $\mathbf{V}^\#_M(\omega)$.

Assumption 4 *The coupling term C_M is $|\cdot|^M_\omega$ -continuous and there exist functions $h^{\alpha\beta} = h^{\beta\alpha} \in L^2(\omega)$ such that for all $\eta \in \mathbf{V}(\omega)$, $\int_\omega h^i \eta_i \sqrt{a}dy = \int_\omega h^{\alpha\beta}\gamma_{\alpha\beta}(\eta)\sqrt{a}dy$ (i.e. the surface density of forces (h^i) is admissible in the sense given in [Luce-Poutous-Thomas, 2007]).*

Remark 3 *For instance, if $\mathbf{f} = (0, 0, 1)$ is admissible, then C_M is $|\cdot|^M_\omega$ -continuous.*

Theorem 1 *Suppose assumptions **1, 2, 3** and **4** are satisfied. There exists ζ in $\mathbf{V}^\#_{M0}(\omega) := \{\eta \in \mathbf{V}^\#_M(\omega), C^\#_M(\eta) = 0\}$ and $\mathbf{u}$ in $\mathbf{V}^\#_M(\Omega)$ such that*

$$\mathbf{u}(\varepsilon) \longrightarrow \mathbf{u} \textit{ in } \mathbf{V}^\#_M(\Omega) \textit{ as } \varepsilon \to 0 \quad \textit{and} \quad \overline{\mathbf{u}(\varepsilon)} \longrightarrow \zeta \textit{ in } \mathbf{V}^\#_M(\omega) \textit{ as } \varepsilon \to 0.$$

and the limit ζ satisfies this variation of the scaled 2D variational problem of a linearly elastic generalized membrane shell of the first kind :

$$\left\{ \begin{array}{l} \zeta \in \mathbf{V}^\#_{M0}(\omega), \forall \eta \in \mathbf{V}^\#_{M0}(\omega) \\ B^\#_M(\zeta, \eta) = L^\#_M(\eta). \end{array} \right. \tag{2}$$

To prove this theorem, we use the same pattern as in [Ciarlet-Lods, 1996] and [Luce-Poutous-Thomas, 2007]. We only develop the proof of the parts where there is a difference due to the orthotropy (part (iii)) or to the coupling (parts $(ii), (v), (vii)$ and $(viii)$).

Proof. Part (ii). We need to add new results to part (ii) : there exists a subsequence still denoted $(\mathbf{u}(\varepsilon))_\varepsilon$ such that

$$\lim_{\varepsilon\to 0} C_M(\overline{\mathbf{u}(\varepsilon)}) = \lim_{\varepsilon\to 0} \int_{\Gamma^-} u_3(\varepsilon)\sqrt{g(\varepsilon)}\, d\Gamma = 0.$$

Combining the uniform positive definiteness of $\mathbf{A}(\varepsilon)$, the positivity of $C_\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon))$, assumption 4, the variational problem (1) and [Luce-Poutous-Thomas, 2007] (ii)'s bounding $|L_\varepsilon(\mathbf{u}(\varepsilon))| \leq c\sqrt{\sum_{i,j} \left\|e_{i\|j}(\mathbf{u}(\varepsilon))\right\|^2_{0,\Omega}}$, we obtain

$$\sum_{i,j} \left\|e_{i\|j}(\mathbf{u}(\varepsilon))\right\|^2_{0,\Omega} \leq cB_\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon)) \leq c(B_\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon)) + C_\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon))) = cL_\varepsilon(\mathbf{u}(\varepsilon)) \leq c\sqrt{\sum_{i,j} \left\|e_{i\|j}(\mathbf{u}(\varepsilon))\right\|^2_{0,\Omega}}.$$

Therefore the sequences $(e_{i\|j}(\mathbf{u}(\varepsilon)))_\varepsilon$ and $(C_\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon)))_\varepsilon$ are bounded. Consequently,

$$\partial_3 u_3(\varepsilon) = \varepsilon e_{3\|3}(\mathbf{u}(\varepsilon)) \longrightarrow 0 \quad in \quad L^2(\Omega) \quad and \quad \lim_{\varepsilon\to 0} \int_{\Gamma^-} u_3(\varepsilon)\sqrt{g(\varepsilon)}\, d\Gamma = \lim_{\varepsilon\to 0} \sqrt{\varepsilon}\sqrt{C_\varepsilon(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon))} = 0.$$

Moreover,

$$\begin{aligned}\lim_{\varepsilon\to 0} C_M(\overline{\mathbf{u}(\varepsilon)}) &= \lim_{\varepsilon\to 0}\int_\omega \overline{u_3(\varepsilon)\sqrt{g(\varepsilon)}}\,dy = \lim_{\varepsilon\to 0}\tfrac{1}{2}\int_\Omega u_3(\varepsilon)\sqrt{g(\varepsilon)}_{x_3=0}\,dx\\ &= \lim_{\varepsilon\to 0}\tfrac{1}{2}\int_{\partial\Omega}(x_3-1)u_3(\varepsilon)\sqrt{g(\varepsilon)}_{x_3=0}\,n_3\,ds - \lim_{\varepsilon\to 0}\tfrac{1}{2}\int_\Omega (x_3-1)\partial_3 u_3(\varepsilon)\sqrt{g(\varepsilon)}_{x_3=0}\,dx.\end{aligned}$$

Hence the result holds since $n_3=0$ on $\partial\omega\times[-1,1]$ and $n_3=-1$ on Γ^-.

Part (iii). With the same technique as in [Ciarlet-Lods, 1996] it becomes : $\quad e_{\alpha\|3}=0, \quad e_{3\|3}=-\dfrac{\lambda_{\alpha 3}}{\lambda_{33}}a^{\alpha\alpha}\,e_{\alpha\|\alpha}$.
Part (v). It is changed into : There exists a constant $c_c\in\mathbb{R}$ such that for all $\eta\in\mathbf{V}(\omega)$,

$$\int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\gamma_{\alpha\beta}(\eta)\sqrt{a}\,dy = \int_\omega h^{\alpha\beta}\gamma_{\alpha\beta}(\eta)\sqrt{a}\,dy - c_c\,C_M(\eta).$$

Indeed, let v_3 be in $H^1(\omega)$ such that $v_3=0$ on γ_0 and $\int_\omega v_3\sqrt{a}\,dy\neq 0$. For $\mathbf{v}=(0,0,v_3)$ (which is in $\mathbf{V}(\Omega)$) we have

$$C_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{v}) = \frac{1}{\varepsilon}\int_{\Gamma^-} u_3(\varepsilon)\sqrt{g(\varepsilon)}\,d\Gamma\int_{\Gamma^-} v_3\sqrt{g(\varepsilon)}\,d\Gamma = L_\varepsilon(\mathbf{v}) - B_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{v}).$$

Let ε approach zero, then

$$L_\varepsilon(\mathbf{v})\longrightarrow L_M(\mathbf{v}),\qquad \int_{\Gamma^-} v_3\sqrt{g(\varepsilon)}\,d\Gamma\longrightarrow C_M(\mathbf{v})\quad\textit{and}\quad B_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{v})\longrightarrow\int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}b_{\alpha\beta}v_3\,dy.$$

These limits are in $\mathbb{R}$, hence there exists $c_c\in\mathbb{R}$ such that $\lim\limits_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Gamma^-} u_3(\varepsilon)\sqrt{g(\varepsilon)}\,d\Gamma = c_c$ and the result holds.

Part (vii). To establish the strong convergency of $(e_{i\|j}(\mathbf{u}(\varepsilon)))_\varepsilon$ towards $e_{i\|j}$ in $L^2(\Omega)$ announced in part (vii) we proceed in two steps. First, we prove that

$$\lim_{\varepsilon\to 0}\sum_{i,j}\|e_{i\|j}(\mathbf{u}(\varepsilon)) - e_{i\|j}\|^2_{0,\Omega}\le c(\lim_{\varepsilon\to 0}L_\varepsilon(\overline{\mathbf{u}(\varepsilon)}) - 2\int_\omega A^{ijkl}(0)\overline{e_{k\|l}}\,\overline{e_{i\|j}}\sqrt{a}\,dy).$$

Indeed, from the uniform positive definiteness of $\mathbf{A}(\varepsilon)$ we have

$$\sum_{i,j}\|e_{i\|j}(\mathbf{u}(\varepsilon)) - e_{i\|j}\|^2_{0,\Omega}\le c(B_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) - 2\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\mathbf{u}(\varepsilon))\,e_{i\|j}\sqrt{g(\varepsilon)}\,dx + \int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}\,e_{i\|j}\sqrt{g(\varepsilon)}\,dx.$$

Thereby, as for all (i,j), $e_{i\|j}=\overline{e_{i\|j}}$ (they are independant of x_3), we have

$$\lim_{\varepsilon\to 0}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\mathbf{u}(\varepsilon))\,e_{i\|j}\sqrt{g(\varepsilon)}\,dx = \lim_{\varepsilon\to 0}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}\,e_{i\|j}\sqrt{g(\varepsilon)}\,dx = 2\int_\omega A^{ijkl}(0)\overline{e_{k\|l}}\,\overline{e_{i\|j}}\sqrt{a}\,dy.$$

Moreover,

$$B_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) = L_\varepsilon(\mathbf{u}(\varepsilon)) - C_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon))$$

and

$$\lim_{\varepsilon\to 0}C_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) = \left(\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Gamma^-} u_3(\varepsilon)\sqrt{g(\varepsilon)}\,d\Gamma\right)\left(\lim_{\varepsilon\to 0}\int_{\Gamma^-} u_3(\varepsilon)\sqrt{g(\varepsilon)}\,d\Gamma\right) = c_c\times 0 = 0.$$

So, if they exist, $\lim\limits_{\varepsilon\to 0}B_\varepsilon(\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) = \lim\limits_{\varepsilon\to 0}L_\varepsilon(\mathbf{u}(\varepsilon))$ and letting ε approach zero we obtain the announced inequality.
Second, letting $\eta=\overline{\mathbf{u}(\varepsilon)}$ and ε approach zero in part (v) we prove that

$$\lim_{\varepsilon\to 0}L_\varepsilon(\mathbf{u}(\varepsilon)) = 2\int_\omega A^{ijkl}(0)\overline{e_{k\|l}}\,\overline{e_{i\|j}}\sqrt{a}\,dy.$$

Indeed, as $\lim_{\varepsilon\to 0} C_M(\overline{\mathbf{u}(\varepsilon)}) = 0$ and $L_\varepsilon(\mathbf{u}(\varepsilon)) = L_\varepsilon(\overline{\mathbf{u}(\varepsilon)})$, we obtain

$$\lim_{\varepsilon\to 0} L_\varepsilon(\mathbf{u}(\varepsilon)) = \lim_{\varepsilon\to 0} \int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\gamma_{\alpha\beta}(\overline{\mathbf{u}(\varepsilon)})\sqrt{a}\,dy.$$

We conclude thanks to the weak convergence of $(\gamma_{\alpha\beta}(\overline{\mathbf{u}(\varepsilon)}))_\varepsilon$ towards $\overline{e_{\alpha\|\beta}}$ and noticing that

$$2\int_\omega A^{ijkl}(0)\overline{e_{k\|l}}\,\overline{e_{i\|j}}\sqrt{a}\,dy = \int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\,\overline{e_{\alpha\|\beta}}\sqrt{a}\,dy.$$

Part $(viii)$. It becomes : the limit $\zeta \in \mathbf{V}_M^{\#}(\omega)$ found in part (vii) is in $\mathbf{V}_{M0}^{\#}(\omega)$ and satisfies the equations

$$B_M^{\#}(\zeta,\eta) = L_M^{\#}(\eta), \quad \text{for all } \eta \in \mathbf{V}_{M0}^{\#}(\omega).$$

With the same technique as in [Ciarlet-Lods, 1996], we prove that ζ, which existence has already been proved, satisfies the ill-posed problem

$$\zeta \in \mathbf{V}_{M0}^{\#}(\omega), \quad B_M^{\#}(\zeta,\eta) = L_M^{\#}(\eta) - c_c\, C_M^{\#}(\eta), \quad \forall \eta \in \mathbf{V}_M^{\#}(\omega).$$

Thereby, it satisfies the announced well posed problem. ■

4 Conclusion

From architecture to aeronautics industry, from anatomy to pneumatics, the scope of application of inflated structures is very wide. This Note is the generalization to any kind of linearly elastic generalized membrane shells of the first kind of results obtained in [Poutous, 2006] when studying the behaviour of the outer envelop of an airship. The generalization has supposed to broaden some assumptions: for instance the continuity in the energy norm of the the coupling term was proved whereas it is assumed in this Note.

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